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AN ALTERNATIVE APPROACH TO AGGREGATE SURPLUS ANALYSIS

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This paper suggests a dual to the many-person applied welfare economics problem with constraints on lump-sum redistribution. The dual has the property of minimizing an aggregator function over individual income transfers. The properties of the aggregator are dependent upon the resource costs of redistribution and not upon the distributional preferences embodied in the social welfare function. An interpretation of the dual problem in terms of consumer surplus is offered.

1. Introduction

Practitioners of applied welfare economics often use unweighted sums of individual consumer surplus changes to evaluate public policy. This is generally seen as an efficiency-oriented technique that does not attempt to reflect judgements about the equity effects of policy. It is widely recognized, of course, that distributional issues cannot be avoided in many policy contexts (e.g. in tax policy evaluation), and in these cases the usual procedure is to employ weighted sums of individual surplus changes.

In this paper we present a different view on aggregating individual surpluses. Beginning with the primal problem of finding a policy that maximizes some Bergson–Samuelson social welfare function (SWF) defined over individual utilities, subject to relevant constraints, we define a dual problem that provides a many-person generalization of the Diamond–McFadden (1974) loss-minimization approach to the optimal tax-applied

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welfare economics problem. Then we show that the minimization of a many-person loss function is equivalent to the maximization of a 'consumer surplus aggregator'. We are able to establish the duality between our primal and dual problems without any aggregation-type restrictions on preferences (e.g. homotheticity or zero income effects). This permits us to conclude, for example, that any solution to a second-best social welfare maximization problem can also be characterized as a solution to a loss minimization or aggregate surplus problem, and conversely.

More specifically, we formulate welfare maximization problems in which the welfare optimizer not only controls tax rates (and the other usual instruments, such as public production), but can also determine a (positive or negative) lump-sum transfer to each household, subject to the constraint that a non-decreasing function defined over individual transfers, called a *redistributive cost function*, be non-positive.[1] The form of this function reflects the 'technology' or costliness of the transfer mechanism. For example, in the important special case where redistribution is costless, the redistributive cost function is a simple unweighted sum of the payments to all households: the constraint that this sum be non-positive then indicates that one more dollar's worth of resources taken from one household permits one more dollar's worth of transfers to be paid out to some other household(s). Alternatively, redistribution may be costly in the sense that only a portion of a dollar taken from one household eventually finds its way into the hands of a recipient. This could occur, perhaps, because of the administrative cost of the redistribution. If money (strictly real resources) taken from a rich household is put into a leaky bucket which will (irretrievably) lose some of its contents by the time it arrives at the poor family's doorstep, or if there is some wasteful melting of the ice cream in the camel caravan that redistributes from rich to poor oases in the desert, then the redistributive cost function will assume a form, perhaps a weighted sum of transfers or some more general non-linear form, that reflects these costs.[2] In the extreme, we arrive at another important polar case, where redistribution is infinitely costly. This reduces essentially to the standard optimal tax problem, in which no redistribution occurs. Here, the redistribution constraint requires that the maximum transfer paid to any household be non-positive, i.e. no amount of resources given up by other households would make it possible to transfer a dollar to any one household.

[1]More generally, the constraint is that the redistributive cost function be less than or equal to some prescribed value not necessarily zero.

[2]On leaky buckets, see Okun (1975). The caravan example is due to Harberger (1978). It should be noted that in their examples, leakage or melting is meant to reflect the efficiency losses due to the redistribution-induced distortion of incentives, as well as any pure resource costs resulting from the transfer mechanism per se. Our analysis treats distortions explicitly and separately, however, so the redistributive cost function must be interpreted more strictly than Okun and Harberger interpret their metaphors.

The redistributive cost function is of critical importance because it becomes the objective function in our loss-minimization dual problem, and it determines the form of the corresponding consumer surplus aggregator or social surplus function (SSF) in the equivalent social surplus maximization problem. The form of the aggregation of individual surpluses, in other words, naturally reflects *not* the distributional preferences embodied in the SWF of the primal problem, but the 'technology' of lump-sum redistribution embodied in the redistributive transfer constraint. This does not imply that distributional preferences, as captured in a SWF, are irrelevant in the formulation of dual surplus maximization problems. Rather, the distributional preferences of the SWF show up in a *constraint* on the dual problem. This constraint specifies the individual reference utility levels, and thus the reference compensated demand curves, with respect to which the individual consumer surpluses are measured.

The remainder of the paper is organized as follows: section 2 presents the general applied welfare economics problem to be considered; section 3 provides a dual characterization of this problem and a formal equivalence result. Section 4 provides a 'consumer surplus' interpretation of the results. In section 5, we summarize our conclusions. Much of the technical argument is relegated to an appendix.

2. The general welfare maximization problem

In order to present the analysis and results in the simplest possible context, we restrict attention to a quite simple and familiar policy problem, namely the determination of an optimal structure of commodity taxes. It is quite straightforward to extend the results to deal with other issues, such as the determination of optimal public expenditures and public production. The analysis also extends directly to the determination of the restricted optimal tax structure, with, for example, some untaxed commodities.

Let W be a Bergson–Samuelson SWF, defined over individual utilities, non-decreasing in each argument and strictly increasing in at least one. The standard optimal tax problem of Diamond and Mirrlees (1971) can be written[3]

[3]The notation is standard. There are H households, $h=1,\ldots,H$, each with utility function $u_h(x_h)$ defined over the net consumption vector x_h. (Factors supplied are negative net consumption.) The normalized vector of consumer prices is q, the expenditure function (denominated in numeraire) is $e_h(q,u_h)$, and the compensated demand function is $x_h(q,u_h)$. $X(q,u)=\sum_h x_h(q,u_h)$, where $u=(u_1,\ldots,u_h)$. $m=(m_1,\ldots,m_H)$ is a vector of lump-sum incomes (measured in numeraire). $e=(e_1,\ldots,e_H)$. Private production is organized competitively, either with constant returns technology or 100 percent profits taxes, so no profits accrue to households. (The results extend to less than 100 percent profits taxes, however.) G is the net-of-government-consumption set, representing feasible net private consumption possibilities.

$$\text{(P)} \qquad \max_{\langle q,u\rangle} W(u)$$

s.t.

$$X(q,u)\in G, \quad e_h(q,u_h)=0, \quad \text{all } h.$$

The second constraint in (P) implies that lump-sum tranfers are ruled out. Alternatively, the second constraint in (P) could be replaced by

$$\sum_h e_h(q,u_h) \leqq 0, \tag{1}$$

allowing for the possibility of lump-sum interpersonal redistribution.

It is implicit in constraint (1) that a unit of numeraire taken from household h permits a transfer of one more unit to household h'. In this sense, (1) characterizes *costless* redistribution. It is imaginable, however, that a unit of numeraire 'shrinks' en route to household h', as in the leaky bucket example given earlier. Then not all points in m-space satisfying (1) will be attainable in the absence of some supply of numeraire, exogenous to the redistributive program, to absorb the transfer costs. As fig. 1 illustrates, the line R^0 shows the frontier of points satisfying (1), attainable via costless

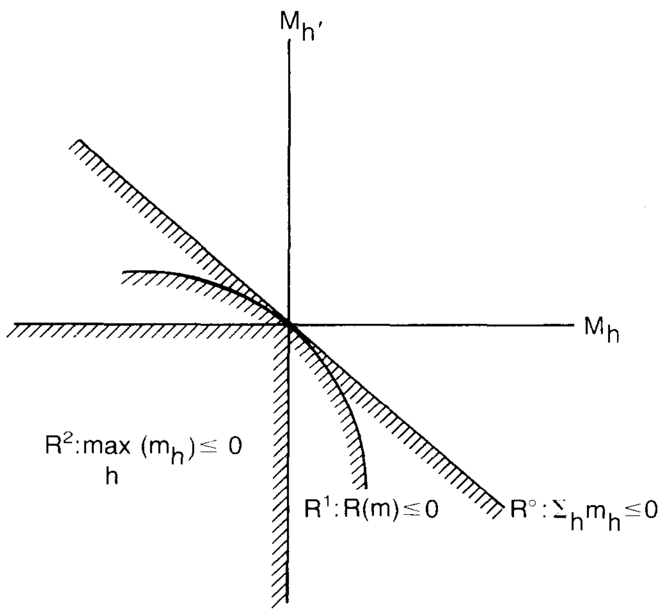

Fig. 1. Redistribution cost functions.

redistribution. When redistribution is costly, the point $m=0$ is still attainable, but any redistribution forces one below R^0, as illustrated for example by the frontier R^1.

Algebraically, any redistributive constraint can be represented by a function $R(m)$ which is constrained to be non-positive, and is non-decreasing in each argument and strictly increasing in at least one. We call $R(m)$ a *redistributive cost function.* $R(m)$ is interpreted as the amount of the numeraire good that must be exogenously supplied to achieve the redistributive program m. $R(0_H)=0$ always. The special case of infinitely costly redistribution, depicted by R^2 in fig. 1, is represented mathematically as

$$R(m)=\max_{\langle h\rangle}\{m_h\}\leqq 0. \tag{2}$$

In general, then, we can write the welfare maximization problem with possibly costly redistribution as (P) with the second constraint replaced by $R[e(q,u)]\leqq 0$, or, with $\bar{r}$ an exogenously fixed scalar, by $R[e(q,u)]\leqq \bar{r}$. Obviously this encompasses (1), (2), and all intermediate cases as well.[4]

3. The dual many-person expenditure (loss) minimization problem

Diamond and McFadden (1974) have stressed the usefulness of the loss-minimization approach to optimal taxation. For example, it permits a very simple derivation of the Ramsey characterization of the optimal tax structure. It also facilitates a consumer surplus interpretation of the optimal tax structure and of the deadweight loss from taxation. A natural question is whether this can generalize to the many-person context. Actually, examples of such generalizations already exist in the literature. Helpman (1978), for example, evaluates tariff policy in terms of an unweighted sum of individual expenditure functions. Is this approach consistent with some primal welfare objective, however? For example, is a policy which minimizes this unweighted sum actually feasible, and is it Pareto efficient?[5] If not, on what basis are we to justify interest in this policy? Moreoever, one might wonder about the equity implications of the unweighted sum approach. How should it be modified to accommodate differing equity viewpoints?

In this section, we address these issues. We begin by observing that a natural dual to problem (P), obtained by an interchange of objective and

[4] $\bar{r}$, if non-zero, can be interpreted as a supply of numeraire (manna), from a source external to the economy (heaven), that increases the amount of resources that can be distributed to households via the determination of the vector m. $\bar{r}$, if negative, represents an exogenously fixed claim on the economy's resources that constrains the determination of lump-sum incomes m. $\bar{r}$ is included purely as a formal device. In most situations of interest $\bar{r}$ will equal zero.

[5] Actually, these questions arise in the single-person case as well. Our results include the single-person economy as a special case.

constraint functions, is:

$$\text{(D)} \qquad \min_{\langle q,u\rangle} R[e(q,u)]$$

s.t.

$$X(q,u) \in G \quad \text{and} \quad W(u) \geqq \bar{w}.$$

First, we note that (D) is in fact a many-person generalization of the Diamond–McFadden problem. In their approach, one chooses consumer prices q to minimize a single consumer's expenditure function $e(q,\bar{u})$ subject to a feasibility constraint $x(q,\bar{u}) \in G$, where the utility level $\bar{u}$ is exogenously fixed. The single-consumer version of (D) is precisely identical to this, except that u is added as a choice variable, subject to the constraint that $u \geqq \bar{u}$. This latter constraint will bind at a solution (under weak assumptions), so that (D) really is equivalent to the Diamond–McFadden problem for single consumers.

In the general many-consumer case, (D) involves minimizing the resource cost of a redistributive program $m = e(q,u)$, subject to technological and government budget constraints, and subject to meeting an exogenously specified level of welfare. With many consumers, the constraint $u \geqq \bar{u}$ naturally generalizes to $W(u) \geqq \bar{w}$, which requires that the outcome in utility space at a solution to (D) must lie on or above a contour of the SWF. In fact, because the functions R and W play a role in (D) quite analogous to e and u in the single-consumer case, one could think of R as a 'social expenditure function'.

To establish a formal relationship between (P)[6] and (D), we require that the SWF W, the redistributive cost function R, and the underlying preferences and technology be well behaved. In addition, we require that constraint qualification conditions be satisfied for both (P) and (D). These assumptions are spelled out in the appendix. There we also state more formally and prove the following key results:

Theorem 1. If (q^*, u^*) *is a solution to* (P), *then* (q^*, u^*) *is a solution to* (D), *for* $\bar{w} = W(u^*)$. *At this solution,* $R[e(q^*, u^*)] = \bar{r}$.

Theorem 2. If $(\tilde{q}, \tilde{u})$ *is a solution to* (D), *then* $(\tilde{q}, \tilde{u})$ *is a solution to* (P) *for* $\bar{r} = R(e(\tilde{q}, \tilde{u})]$. *At this solution,* $W(\tilde{u}) = \bar{w}$.

These results have the following implications. First, we observe that if (q^*, u^*) solves (P) with, say, $\bar{r} = 0$, and if we set the parameter $\bar{w}$ in (D) equal

[6](P) in this section is understood to incorporate the general constraint $R[e(q,u)] \leqq \bar{r}$.

to $W(u^*)$, then any solution to (D) also solves (P). The circularity of this argument obviously presents a problem for actually solving (P) via the dual, but nonetheless it establishes the fact that there exists a parametric specification for the dual which solves any primal, and the dual cannot therefore be dismissed as ad hoc or contradictory to the principle of welfare maximization. This conclusion is quite adequate for some purposes. For example, as Diamond and McFadden emphasize, it is quite simple to derive the Ramsey optimal tax formula or the other necessary conditions for optimal policy from (D). The fact that this formula characterizes a true utility-maximizing policy only if $\bar{w}$ is the utility level at the optimum, while problematic for a policymaker trying to find the optimum, is not a weakness arising from the choice of the dual approach: Precisely the same problem besets the analyst beginning with the primal. This weakness is inherent in the Ramsey formula itself, which requires information about the compensated demand curves obtaining at a second-best optimum.

Second, the dual approach does offer at least *some* information about whether a given policy is welfare maximizing: the solution $(\tilde{q}, \tilde{u})$ of the dual is a solution to the primal (P) with $\bar{r}=0$ if and only if $R[e(\tilde{q}, \tilde{u})]=0$. One could imagine using this fact to grope one's way to a solution to (P), by increasing $\bar{w}$ whenever $R<0$ at a solution to (D) and conversely if $R>0$. Whether this would be computationally efficient would of course depend on the relative ease of solving (D) compared to (P).

What is most striking about theorems 1 and 2 is their implications for the problem of equity. Note that the form of the objective function in (D) is determined not by the equity judgments embodied in the SWF W, but rather the aggregate loss measure of the dual corresponds to the redistributive cost function of the primal. If costless lump-sum transfers are feasible, then R is a simple unweighted sum irrespective of equity concerns. In the other natural polar case, where no redistribution is feasible, the dual objective is to minimize the maximum expenditure function. A weighted sum may emerge when redistributive costs are neither zero nor infinite. In any case, the form R reflects the redistributive technology. Equity concerns are nonetheless central to the dual approach. They appear, however, in the welfare constraint $W(u) \geqq \bar{w}$ which restricts the set of feasible utility levels, rather than in the objective function.

Baldly stated, our conclusion that the duality between welfare maximization and loss minimization requires that the individual expenditure functions be evaluated at utility levels satisfying the welfare constraint in (D) is not really surprising. If u^* is the vector of utilities at a solution to (P) and u^0 is some other arbitrary vector, one could hardly expect to characterize the policy leading to u^* in terms, say, of compensated demand elasticities evaluated at $u=u^0$ — the demand functions of consumers might, after all, be quite different at u^0. In the literature, however, relatively little attention is devoted to the determination of the 'reference' utility levels, while equity judgments are supposed to be reflected in the form of the loss function R. See, for example, Tresch (1981, pp. 85–87, 350–353) for a clear statement of the usual approach and the reasons why duality results such as those given above do not obtain in this case.

Finally, let us note that the dual problem (D) suggests a natural deadweight loss interpretation. Imagine solving (D) for a given welfare level $\bar{w}$ and instrument set. This yields a value of the redistribution function $R=R'$. A restriction on the set of instruments or feasible set G, but with the *same* welfare level $\bar{w}$, would give a new and higher optimal value for $R=R''$. The difference $R''-R'$ might be called the 'deadweight loss' implicit in the restriction of instruments — for example, fixed taxes or tariffs. The loss can be interpreted as the amount of exogenous numeraire income that would be required to be injected into the economy, upon introducing the restrictions on the instrument set, in order to maintain the initial level of social welfare. It is similar in some respects to the Debreu (1951) coefficient of resource utilization in that resources are hypothetically injected or withdrawn from the economy. In this case, however, social welfare is held constant and redistribution is not assumed to be costless.

4. Equivalence of welfare and surplus maximization

We now reformulate the loss-minimization framework of (D) in terms of surplus maximization. We begin by defining a measure of individual consumer's surplus change associated with a change in consumer prices, and then discuss the aggregation of these individual changes for overall policy evaluation.

First, for household h we define an *individual surplus* showing the change in consumer's surplus between an initial reference price vector q^0 and a comparison price q according to

$$s_h(q, q^0, u_h) = e_h(q^0, u_h) - e_h(q, u_h). \tag{3}$$

This individual surplus will be recognized as the Hicks compensating variation for a price change from q^0 to q with money income constant. It is important to note that this surplus change measure depends not only on the price vectors to be compared, but on the utility level u_h at which the expenditure function is evaluated. s_h is the negative of the sum (across commodities) of the areas to the left of the compensated demand curves between the initial and comparison prices.

Next, we define a class of functions over translates of the individual surplus functions. Specifically, a *social surplus function* (SSF) is a function $S\colon \mathbb{R}_H \to \mathbb{R}$ defined over the vector $s(q, q^0, u) - e(q^0, u)$, weakly increasing in each argument. Note that the translation of the $s(q,q^0,u)$ vector by $e(q^0,u)$ depends on the reference prices q^0 and on the utility level u.

By (3), $s(q,q^0,u)-e(q^0,u)=-e(q,u)$. Thus, given a redistributive cost function $R:\mathbb{R}_H\to\mathbb{R}$, we may define the *associated SSF* $S(s[q,q^0,u]-e[q^0,u])\equiv -R(-s[q,q^0,u]+e[q^0,u])$, and conclude immediately that choosing (q,u) subject to constraints to minimize $R(e[q,u])$ is equivalent to maximizing $S(s[q,q^0,u]-e[q^0,u])$ if S is the SSF associated with R. Similarly, given an SSF$S;\mathbb{R}_H\to\mathbb{R}_1$, we may define the *associated redistributive cost function* $R(e[q,u])\equiv -S(-e[q,u])$, and conclude that maximizing S is equivalent to minimizing the associated R. By theorems 1 and 2, therefore, we also have:

Corollary 1. If (q^*,u^*) *solve* (P), *then* (q^*,u^*) *solves*

$$\text{(S)}\qquad \max_{\langle q,u\rangle} S(s[q,q^0,u]-e[q^0,u])$$

$$\text{s.t.}$$

$$X(q,u)\in G \quad \textit{and} \quad W(u)\geqq \bar{w}.$$

Corollary 2. Let $(\tilde{q},\tilde{u})$ *solve* (S). *Let R be the redistributive cost function associated with S, and define* $\bar{r}=R(e[\tilde{q},\tilde{u}])=S(s[\tilde{q},q^0,\tilde{u}])$. *Then* $(q,\tilde{u})$ *solve* (P) *relative to R,* $\bar{r}$.

Corollary 1 asserts that if the reference utility levels are chosen 'correctly', the vector q will be welfare optimal if it maximizes an appropriate SSF. Note that with q^0 and u fixed, an SSF can be interpreted simply as a weakly increasing function of the individual surplus functions. It is thus appropriate to refer to (S) as a problem of 'social surplus maximization'. By corollary 2, a social surplus maximizing policy is also social welfare maximizing, given the associated redistributive cost function and net increment to social resources, $\bar{r}$.

By corollaries 1 and 2, there is an intimate interrelationship between social welfare and social surplus maximization, just as theorems 1 and 2 have demonstrated for welfare maximization and aggregate loss minimization. In fact, much of the interpretive discussion of section 3 carries over directly here, and we need not repeat it. Let us simply summarize some of the main implications. First, every solution to a welfare maximization problem can be equivalently described as a solution to a dual surplus maximization problem. Thus, corollary 1 can be used to characterize a solution to (P) in an interesting way. For instance, a solution to an optimal tax problem, given that costless redistribution is possible, must maximize the unweighted sum of individual surpluses, a result that is perhaps more readily interpretable, though actually equivalent to, the famous Ramsey equi-proportionate reduction in demand characterization. Indeed, we can use corollary 1 to provide a check on the (constrained) Pareto efficiency of any existing policy. Suppose (q^0, u^0) is some initial equilibrium. If q^0 does not solve (S), with q^0 itself taken as the reference price vector and $\bar{u} = u^0$ — that is, if it is possible to increase the SSF with alternative prices, measuring individual surpluses relative to compensated demand curves corresponding to *existing* equilibrium utility levels — then the existing policy is not welfare maximizing for *any* SWF, and it must therefore be (constrained) Pareto inefficient.

Second, by corollary 2 there does exist a specification of a surplus maximization approach to policy evaluation that can result in Pareto efficient outcomes, and that is equitable in the sense of selecting any Pareto-efficient outcome that one might desire. Thus, without imposing any aggregation-type restrictions on preferences, we see that well-defined aggregate surplus functions can be constructed and used as objective functions in maximization procedures to yield efficient, unbiased policies.

Third, it is at the same time clear that not all surplus-maximization exercises that one might consider will have the above properties. For example, if utility levels are specified in some arbitrary fashion, maximization of aggregate surplus defined in terms of the associated arbitrary compensated demand curves will generally result in policies that are either infeasible, inefficient, or inequitable.

Moreover, the SSFs in our dual problem (S) reflect the constraints on the possibility for redistribution, not ethical judgments. To see this clearly, consider the two polar special cases where distribution is either costless or completely infeasible. In the first case, $R(e[q,u]) = \sum_h e_h(q, u_h)$, and the associated SSF is $\sum_h (s_h[q, q^0, u_h] - e_h[q^0, u_h])$. Corollary 1 shows that maximization of an unweighted sum of individual surplus is equivalent to maximization of social welfare in this case, *regardless of the form of the SWF*. The result that an unweighted sum of individual surpluses is to be maximized when costless lump-sum distribution is feasible is rather intuitive.

The second case perhaps more surprising. When distribution is not feasible, $R(e[q,u]) = \max\{e_h(q,u)\}$, so that to maximize the associated SSF as in problem (S) is to $\max_{\langle q,u\rangle} \min_{\langle h\rangle} \{s_h(q, q^0, u_h) - e(q^0, u_h)\}$. That is, the objective is to maximize the minimum of the (translated) individual surplus functions. The SSF assumes the maximin form with this redistributive 'technology' regardless of the underlying SWF.

5. Conclusion

We have shown here that social surplus functions can be constructed and used in maximization problems that are precisely equivalent to the usual social welfare maximization exercises. Fundamentally, therefore, neither is superior to, or more operational than, the other. Of course, in any application one method may be more convenient than another, so it is useful to know that either may be used.

We would emphasize, however, that our version of the surplus maximization problem differs from the 'distributional weights' approach often found in the literature. The conventional interpretation of weighted consumer surplus is that it approximates an indirect social welfare function, as for example in Feldstein (1972). On the other hand, Blackorby and Donaldson (1982) have demonstrated that for weighted consumer surplus to be a globally exact indirect social welfare function requires quite restrictive assumptions on individual preferences and the form of the SWF. The aggregation of consumer surplus proposed in this paper stems from a quite different perspective of looking at the 'dual' to the conventional welfare economics problem. The aggregator reflects the potential, or lack of it, for pure lump-sum redistribution. The aggregator may vary from an unweighted sum to a maximin form. The marginal weight attached to person h is the marginal resource cost of transferring dollar to h.[7] In this framework distributional concerns are met by an appropriate choice of reference utility levels, not by choosing individual welfare weights.

Appendix

To prove theorems 1 and 2, we impose the following assumptions:

A.1. $W(u)$: W is an individualistic Bergson–Samuelson SWF, continuous, non-decreasing and increasing for strong Pareto improvements, i.e. if $u' \gg u$, then $W(u') > W(u)$, where $u = (u^1, \ldots, u^H) \in \mathbb{R}^H$.

A.2. $X(q,u)$: The compensated demand correspondence is jointly continuous in (q,u), derived from H utility maximizing consumers.

A.3. G: The feasible second-best production set for aggregate compensated demand vectors is a non-empty, closed subset of $\mathbb{R}^n$.

A.4. e^h: The expenditure functions are jointly continuous in (q,u^h), concave and linear homogeneous in q, and strictly increasing in u^h.

[7] This does not however mean that a 'weighted sum' type of SSF is never appropriate. Such an SSF arises whenever the redistribution technology is of the appropriate form. For instance, suppose a two-person economy such that a one-unit transfer from household 1 to 2 results in one-half of a unit actually 'arriving' at 2, with half of the unit disappearing 'in transit'. Then, for this two-person society, we would have $R(e) = e_1 + 2e_2$ (at least for $e_1 < 0 < e_2$), and the associated SSF, given (q^0, u), would be $S = [s_1(q, q^0, u_1) - e_1(q^0, u_1) - e_1(q^0, u_1)] + 2[s_2(q, q^0, u_2) - e_2(q^0, u_2)]$. Of course, this R also implies that a one-unit transfer from 2 results in two units received by 1, which is hardly imaginable; thus the above specification of R should be taken to refer to the range where 1 is making transfers to 2, which, we might suppose, is the 'relevant range' for the problem at hand.

A.5. $R(e)$: The redistribution function $R:\mathbb{R}^H \to \mathbb{R}$ is continuous, non-decreasing in e and, if $e' \gg e$, then $R(e') > R(e)$.

Both (P) and (D) are second-best problems. In order for these to be 'true' second-best problems, a regularity condition, or 'constraint qualification', analogous to Diamond and Mirrlees' (1971) 'Pareto improving price changes' condition, is required.

Assumption Z.1. At any solution $(\bar{q},\bar{u})$ to (P) or (D), if one exists, for some $\delta > 0$, and all $\varepsilon, \delta > \varepsilon > 0$, for any u such that $\|\bar{u}-u\| < \varepsilon$, there exists a q such that $X(q,u) \in G$.

Assumption Z.2. At any solution $(\bar{q},\bar{u})$ to (P) or (D), for all δ sufficiently small there exists some $u \ll \bar{u} \|u-\bar{u}\| < \delta$, such that $X(\bar{q},u) \in G$.

Remark 1. Assumption Z.1 implies that if one were to relax the constraint $R(e) \leqq \bar{r}$ in (P) at the second-best optimum, it would be possible to raise welfare, i.e. this constraint will be binding at any solution to (P).

Remark 2. Assumption Z.2 is also important for similar reasons, although it could be weakened. Since $R(e)$ is increasing in u if Z.2 *did not hold* it might be possible to have a solution to (D) in which the welfare constraint $W(u) \geqq \bar{w}$ does not bind.

Fig. 2 illustrates a failure of Z.2. Due to inferiority of one good the expansion path at second-best prices $\bar{q}$ actually cuts outside of G for lower utility. Thus, the duality of the two problems is destroyed unless $\bar{w} = W(\bar{u})$. The possibility in problem (D) that, at second-best prices, relaxation of the welfare constraint leads to an increase in the amount of redistribution required, is eliminated by Z.2. A sufficient condition which imples Z.2 is that all goods be normal, although this is actually stronger than necessary. Another sufficient condition is that an income transfer to every household at constant prices increase the total taxes paid by every household.

We now formally restate and prove the two main theorems.

Theorem 1. Given Z.1, *if* (q^*,u^*) *is a solution to* (P), *then* (q^*,u^*) *is a solution to* (D), *for* $\bar{w} \equiv W(u^*)$. *At this solution,* $R[e(q^*,u^*)] = \bar{r}$.

Proof. Suppose to the contrary $\exists (q',u')$ which is a solution for (D) and $R(e') < R(e^*)$. Consequently:

(i) $R(e') < R(e^*) = \bar{r}$;
(ii) $X(q',u') \in G$;
(iii) $W(u') \geqq W(u^*)$.

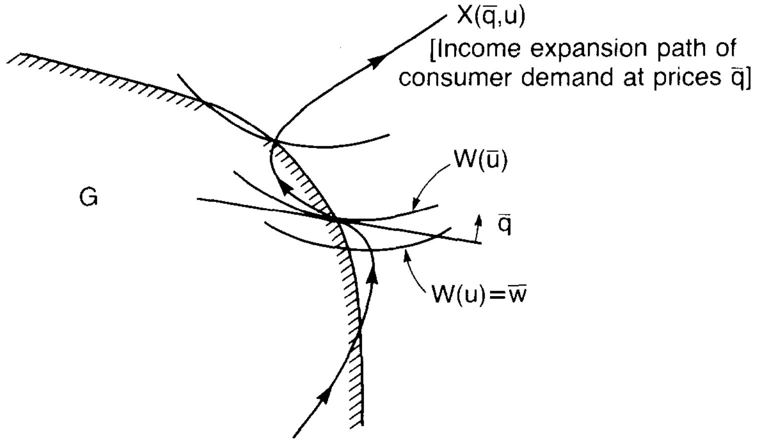

Fig. 2. Violation of Z.2.

Given a $u \gg u'$, but sufficiently close to u', we can find a q close to q' such that

(iv) $R(e) < R(e^*) = \bar{r}$ by continuity of R;
(v) $X(q,u) \in G$ by assumption Z.1;
(vi) $W(u) > W(u') \geqq W(u^*)$ by construction.

Hence (q,u) is feasible for (P) and yields a solution value greater than $W(u^*)$. But this contradicts the assumption that (q^*,u^*) is a solution to (P). Q.E.D.

Theorem 2. Given Z.2, *if* $(\tilde{q},\tilde{u})$ *is a solution to* (D), *then* $(\tilde{q},\tilde{u})$ *is a solution to* (P) *for* $\bar{r} = R(\tilde{e}) \equiv R(e[\tilde{q},\tilde{u}])$.

Proof. Suppose to the contrary $\exists(q',u')$ which is a solution for (P) and $W(u') > W(\tilde{u})$. Then

(i) $W(u') > W(\tilde{u}) \geqq \bar{w}$;
(ii) $X(q',u') \in G$;
(iii) $R(e') \leqq R(\tilde{e}) = \bar{r}$.

By Z.2 choose a $\hat{u}$ close to $\hat{u} \ll u'$, such that

(iv) $R[e(q',\hat{u})] < R(e') \leqq \bar{r}$ by A.5;
(v) $X(q',\hat{u}) \in G$ by Z.2;
(vi) $W(\hat{u}) > W(\tilde{u}) \geqq \bar{w}$ by continuity of W and (i).

Consequently $(q',\hat{u})$ is feasible for (D) and yields a lower value for $R(\cdot)$, contradicting the assumption that $(\tilde{q},\tilde{u})$ is a solution to (D). Q.E.D.

Putting theorems 1 and 2 together, there is a symmetric relationship between the two problems if both Z.1 and Z.2 hold. Note that if Z.1 fails, it is still true that a solution to (D) will be a solution to (P), but not conversely. If Z.2 fails but not Z.1, a solution to (D) will be a solution to (P), but not conversely.

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